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On Geometric Constructions of third order methods for multiple roots of nonlinear equations

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Abstract

The object of the present work is to give geometric constructions of third order methods obtained from the Traub-Gander class for multiple roots when the multiplicity is known. Also, new families of methods for solving nonlinear equations of third order with multiple roots and their geometric construction are presented. Finally, several numerical examples to show the performance of some members of the families mentioned are presented.

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1 Introduction

Iterative methods are necessary usually for solving nonlinear equations f(x) = 0, with $f : \mathbb{C} \to \mathbb{C}$. Several good methods exist in the literature: Newton, Chebyshev, Halley, Super-Halley, Ostrowski, and Euler methods between others ([1]-[3]). In addition, several families of methods have been presented by several authors, where some belong to the class given by Gander in [4], among which we can mention: Chebyshev-Halley type methods and the $\theta - C$ family. In previous papers, the study of the geometric constructions of various iterative methods for simple roots is done, see ([5]-[10]) for example. In the literature, we can also find adaptations of these methods to the case when the nonlinear equation has multiples roots with multiplicity *m* known. Thus, the modified Newton method due to Schröder [11] is given by

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots.$$
 (1.1)

Some adaptations for the calculation of multiple roots with known multiplicity of the methods and families for simple roots mentioned above will be given in section 3. Recently some geometric constructions of two variants of Newton's method to solving nonlinear equations with multiple roots have been presented in [12]. The author does not know of the existence of literature related to the geometric constructions of methods and/or families of methods with third order of convergence to solve nonlinear equations with multiple roots. So, in section 2 we give geometric constructions of elements of the class of Traub-Gander to solve nonlinear equations with multiple roots when the multiplicity is known. Then, in section 3 new families of methods based on the class of Traub-Gander and its geometric constructions are shown. In section 4, numerical examples to demonstrate the performance of some members of the families mentioned are presented. By last final remarks are presented.

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2 Geometric Constructions

In this section, we present a geometric construction of third-order methods to solve nonlinear equations with roots multiple when the multiplicity is known based on the Traub-Gander's class given in [13]. If H(0) = 1, $\dot{H}(0) = \frac{1}{2}$ and $|\ddot{H}(0)| < \infty$ then

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} H\left(1 - m + mL_f(x_n)\right); \quad with \quad L_f(x) = \frac{f(x)f''(x)}{\left[f'(x)\right]^2}$$
(2.2)

is of third order and the error is

$$e_{n+1} = K(\alpha)e_n^3 + O(e_n^4)$$

where the asymptotic error constant $K(\alpha)$ is given by

$$K(\alpha) = \frac{1}{2m^2} \left[\left(m + 3 - 4\ddot{H}(0) \right) A_1^2 - 2mA_2 \right],$$
(2.3)

where $A_k = \frac{m!}{(m+k)!} \frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)}, k = 1, 2$ and α is a root of f with multiplicity m.

Now, we present a result that allows us to obtain a curve from which any iterative method or family of third-order methods given by the Traub-Gander class can be constructed.

Theorem 2.1. Let α a multiple zero of f with multiplicity m and H any function with H(0) = 1, $\dot{H}(0) = \frac{1}{2}$ and $|\ddot{H}(0)| < \infty$. The iteration $x_{n+1} = G(x_n)$, with $G(x) = x - m \frac{f(x)}{f'(x)} H(1 - m + mL_f(x))$ where $L_f(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$, can be built from the curve that has by equation

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y(x)}{f(x_n)}} \right) H\left(\left(1 - m + mL_f(x_n) \right) \left(1 - \sqrt[m]{\frac{y(x)}{f(x_n)}} \right) \right)$$
(2.4)

and this complies with the following tangency conditions

1. $y(x_n) = f(x_n)$ 2. $y'(x_n) = f'(x_n)$ 3. $y''(x_n) = f''(x_n)$

Proof. If $x = x_{n+1}$ and $y(x_{n+1}) = 0$ are used in (2.4), then $x_{n+1} = G(x_n)$. By replacing $x = x_n$ in the equation (2.4) then $y(x_n) = f(x_n)$ is obtained. Now, let the constant $A = 1 - m + mL_f(x_n)$ and $g(y(x)) = A\left(1 - \sqrt[m]{\frac{y(x)}{f(x_n)}}\right)$ then

$$g(y(x_n)) = 0, \quad g'(y(x_n)) = -\frac{A}{m} \frac{f'(x_n)}{f(x_n)} \quad and$$
 (2.5)

$$g''(y(x_n)) = -\frac{A}{m} \left(\frac{1-m}{m} \left(\frac{f'(x_n)}{f(x_n)} \right)^2 + \frac{f''(x_n)}{f(x_n)} \right).$$
(2.6)

To simplify, we writing (2.4) in the following form

$$\frac{A}{m}\frac{f'(x_n)}{f(x_n)}(x - x_n) = -g(y(x))H(g(y(x)))$$
(2.7)

Now deriving (2.7) twice with respect to the variable x and \dot{H} represents the derivative of H respect to g we obtain

$$\frac{A}{m}\frac{f'(x_n)}{f(x_n)} = -g'(y(x))H(g(y(x))) - g(y(x))\dot{H}(g(y(x)))g'(y(x))$$
(2.8)

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and

$$0 = -g''(y(x))H(g(y(x))) - 2\dot{H}(g(y(x)))\left[g'(y(x))\right]^2 - g(y(x))\left[g'(y(x))H(g(y(x)))\right]'$$
(2.9)

Evaluating $x = x_n$ in (2.8) and (2.9) and using equations (2.5) and (2.6) the other tangency conditions are obtained.

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If m = 1, the case of the geometric construction presented in [10] is obtained

3 Examples

In this section, we present several examples of families of third order methods belonging to the of Traub-Gander's class (2.2) to calculate multiple roots when the multiplicity *m* is known based on the weight functions *H* given by the author in [10]. In the iteration equations, x_0 is given and $n = 0, 1, 2, \cdots$. Also, the asymptotic error constant $K(\alpha)$ calculated using (2.3) for the given families and/or methods are presented. All the curves that are shown in this section comply with the conditions of tangency mentioned in the theorem 2.1. If we do the substitution $x = x_{n+1}$ and y = 0 in these curves, then the respective iteration equations are obtained.

3.1 Hansen-Patrick's type family to multiple roots [14]

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Let the weight function *H* be given by

$$H(t) = \frac{a+1}{a+\sqrt{1-(a+1)t}}$$
 $(a \neq -1)$ where $\ddot{H}(0) = \frac{a+3}{4}$

So, the equation of iteration is

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \frac{(a+1)}{\left(a + \sqrt{1 - (a+1)\left(1 - m + mL_f(x_n)\right)}\right)}$$

with $K(\alpha) = \frac{1}{2m^2} ((m-a)A_1^2 - 2mA_2)$, which can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \frac{a+1}{a + \sqrt{1 - (a+1)\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)}}$$
(3.10)

Among the particular cases of this family can be mentioned

• When a = 0, we obtain the Ostrowski-type method to multiple roots

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \frac{1}{\sqrt{m\left(1 - L_f(x_n)\right)}}$$
(3.11)

with $K(\alpha) = \frac{1}{2m} (A_1^2 - 2A_2).$

• When a = 1, we obtain the Euler-type method to multiple roots (in this case $\ddot{H}(0) = 1$)

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \frac{2}{\left(1 + \sqrt{2m\left(1 - L_f(x_n)\right) - 1}\right)}$$
(3.12)

with $K(\alpha) = \frac{1}{2m^2} ((m-1)A_1^2 - 2mA_2).$

• When a = m, we obtain (in this case $\ddot{H}(0) = \frac{m+3}{4}$)

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$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \frac{m+1}{m + \sqrt{m^2 - m(m+1)L_f(x_n)}}$$
(3.13)

with $K(\alpha) = -\frac{A_2}{m}$.

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3.2 Chebyshev-Halley-type family to multiple roots [15]

Let the weight function H given by

$$H(t) = 1 + \frac{t}{2(1 - At)} \Rightarrow \ddot{H}(0) = A$$

the equation of iteration to the Chebyshev-Halley's type family to multiple roots is

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 + \frac{1 - m + mL_f(x_n)}{2\left(1 - A\left(1 - m + mL_f(x_n)\right)\right)} \right)$$
(3.14)

and $K(\alpha) = \frac{1}{2m^2} ((m+3-4\alpha)A_1^2 - 2mA_2)$, which can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \left(1 + \frac{\left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)}{2\left(1 - A\left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)\right)}\right)$$

Among the particular cases of this family can be mentioned

• When A = 0, we obtain the Chebyshev-type method to multiple roots

$$x_{n+1} = x_n - \frac{m}{2} \frac{f(x_n)}{f'(x_n)} \left(3 - m + mL_f(x_n)\right)$$
(3.15)

with $K(\alpha) = \frac{1}{2m^2} [(m+3)A_1^2 - 2mA_2].$

• When $A = \frac{1}{2}$, we obtain the Halley-type method to multiple roots

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{2m}{1 + m - mL_f(x_n)}$$
(3.16)

with $K(\alpha) = \frac{1}{2m^2} \left[(m+1)A_1^2 - 2mA_2 \right].$

• When A = 1, we obtain the Super-Halley-type method to multiple roots

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{1 + m - mL_f(x_n)}{2\left(1 - L_f(x_n)\right)}$$
(3.17)

with $K(\alpha) = \frac{1}{2m^2} [(m-1)A_1^2 - 2mA_2].$

• When $A = \frac{m+3}{4}$, we obtain the method

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \frac{m^2 + 2m + 9 + m(m+5)L_f(x_n)}{m^2 + 4m + 7 - m(m+3)L_f(x_n)}$$
(3.18)

with $K(\alpha) = -\frac{A_2}{m}$.

3.3 θ -C type family to multiple roots [16]

Let the weight function H given by

$$H(t) = 1 + \frac{1}{2} \frac{t}{(1 - \theta t)} + Ct^2 \Rightarrow \ddot{H}(0) = \theta + 2C$$

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So, the equation of iteration is

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 + \frac{1 - m + mL_f(x_n)}{2\left(1 - \theta\left(1 - m + mL_f(x_n)\right)\right)} + C\left(1 - m + mL_f(x_n)\right)^2 \right)$$

and $K(\alpha) = \frac{1}{2m^2} \left[(m+3-4(\theta+2C))A_1^2 - 2mA_2 \right]$ which can be constructed from curve

$$\begin{aligned} x &= x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right) \left(1 + \frac{1}{2} \frac{\left(1 - m + mL_f(x_n) \right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right)}{1 - \theta \left(1 - m + mL_f(x_n) \right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right)} \\ &+ C \left(1 - m + mL_f(x_n) \right)^2 \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right)^2 \right) \end{aligned}$$

Among the particular cases of this family can be mentioned

• When $\theta = 0$, we obtain the *C*-type family to multiple roots (in this case $\ddot{H}(0) = 2C$)

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{3 - m + mL_f(x_n)}{2} + C \left(1 - m + mL_f(x_n) \right)^2 \right)$$
(3.19)

and $K(\alpha) = \frac{1}{2m^2} \left[(m+3-8C)A_1^2 - 2mA_2 \right]$

• When $\theta = 0$ and $C = \frac{m+3}{8}$, we obtain the following method

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{3 - m + mL_f(x_n)}{2} + \frac{m+3}{8} \left(1 - m + mL_f(x_n) \right)^2 \right)$$
(3.20)

and $K(\alpha) = -\frac{A_2}{m}$

- When C = 0, the Chebyshev-Halley-type family to multiple roots (3.14) is obtained.
- When $C = \frac{m+3-4\theta}{8}$ the following family is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 + \frac{1 - m + mL_f(x_n)}{2\left(1 - \theta\left(1 - m + mL_f(x_n)\right)\right)} + \frac{m + 3 - 4\theta}{8} \left(1 - m + mL_f(x_n)\right)^2 \right)$$

with $K(\alpha) = -\frac{A_2}{m}$

3.4 A convex combination of the Halley and Chebyshev methods to multiple roots Let the weight function H given by

$$H(t) = \frac{A}{\left(1 - \frac{1}{2}t\right)} + \left(1 - A\right)\left(1 + \frac{1}{2}t\right) \Rightarrow \ddot{H}(0) = \frac{A}{2}$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{2A}{1 + m - mL_f(x_n)} + \frac{(1 - A)}{2} \left(3 - m + mL_f(x_n) \right) \right)$$

and $K(\alpha) = \frac{1}{2m^2} \left[(m+3-2A)A_1^2 - 2mA_2 \right]$. This iteration equation can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right) \left(\frac{A}{1 - \frac{1}{2} \left(1 - m + mL_f(x_n) \right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right)} + (1 - A) \left(1 + \frac{1}{2} \left(1 - m + mL_f(x_n) \right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right) \right) \right)$$

Among the particular cases of this family can be mentioned

- When A = 0 we obtain the Chebyshev-type method to multiple roots (3.15).
- When A = 1 we obtain the Halley-type method to multiple roots (3.16).
- When A = 2 the following method is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{4}{1 + m - mL_f(x_n)} + -\frac{1}{2} \left(3 - m + mL_f(x_n) \right) \right)$$
(3.21)

with $K(\alpha) = \frac{1}{2m^2} \left[(m-1)A_1^2 - 2mA_2 \right]$

• When $A = \frac{m+3}{2}$ the following method is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{m+3}{1+m-mL_f(x_n)} - \frac{(m+1)}{4} \left(3 - m + mL_f(x_n) \right) \right)$$
(3.22)

with $K(\alpha) = -\frac{A_2}{m}$

3.5 A convex combination of the Newton's method and one Newton-Halley type method to multiple roots Let the weight function *H* given by

$$H(t) = A + \frac{2(A-1)^2}{2(1-A)-t}; \Rightarrow \ddot{H}(0) = \frac{1}{2(1-A)}; (A \neq 1)$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(A + \frac{2(A-1)^2}{2(1-A) - (1-m+mL_f(x_n))} \right); (A \neq 1)$$

and $K(\alpha) = \frac{1}{2m^2} \left[\left(\frac{(m+3)(1-A)-2}{1-A} \right) A_1^2 - 2mA_2 \right]$. This iteration equation can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \left(A + \frac{2(A-1)^2}{2(1-A) - \left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)}\right)$$

Among the particular cases of this family can be mentioned

- When A = 0 we obtain the Halley-type method to multiple roots (3.16).
- When $A = \frac{1}{2}$ we obtain the Super-Halley-type method to multiple roots (3.17).
- When $A = \frac{m+5}{m+3}$ the following method is obtained

$$x_{n+1} = x_n - \frac{m}{(m+3)} \frac{f(x_n)}{f'(x_n)} \left(m + 5 - \frac{4}{4 + (m+3)(1 - m + mL_f(x_n))} \right)$$
(3.23)

with $K(\alpha) = -\frac{A_2}{m}$.



3.6 A new family of Chebyshev-Halley type methods to multiple roots

Let the weight function H given by

 x_n

$$H(t) = \frac{1}{1 - At} + \left(\frac{1}{2} - A\right)t; \Rightarrow \ddot{H}(0) = 2A^2$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{1 - A(1 - m + mL_f(x_n))} + \left(\frac{1}{2} - A\right) (1 - m + mL_f(x_n)) \right)$$

and $K(\alpha) = \frac{1}{2m^2} \left[\left(m + 3 - 8A^2 \right) A_1^2 - 2mA_2 \right]$, which can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \left(\frac{1}{1 - A\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)} + \left(\frac{1}{2} - A\right)\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)\right)$$

Among the particular cases of this family can be mentioned

- When A = 0 we obtain the Chebyshev-type method to multiple roots (3.15).
- When $A = \frac{1}{2}$ we obtain the Halley-type method to multiple roots (3.16).
- When $A = \sqrt{\frac{m+3}{8}}$ the following method is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{1 - \sqrt{\frac{m+3}{8}}(1 - m + mL_f(x_n))} + \left(\frac{1}{2} - \sqrt{\frac{m+3}{8}}\right)(1 - m + mL_f(x_n)) \right)$$
(3.24)

Similarly, when $A = -\sqrt{\frac{m+3}{8}}$ the following method is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{1 + \sqrt{\frac{m+3}{8}}(1 - m + mL_f(x_n))} + \left(\frac{1}{2} + \sqrt{\frac{m+3}{8}}\right)(1 - m + mL_f(x_n)) \right)$$
(3.25)

with the same $K(\alpha) = -\frac{A_2}{m}$

3.7 A single parameter family of third order method to multiple roots

Let the weight function H given by

$$H(t) = \frac{1}{(1 - At)\left(1 - \left(\frac{1}{2} - A\right)t\right)}; \Rightarrow \ddot{H}(0) = \frac{4A^2 - 2A + 1}{2}$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{\left(1 - A(1 - m + mL_f(x_n))\right) \left(1 - \left(\frac{1}{2} - A\right)(1 - m + mL_f(x_n))\right)} \right)$$

with $K(\alpha) = \frac{1}{2m^2} \left[\left(m + 1 - 8A^2 + 4A \right) A_1^2 - 2mA_2 \right]$, which can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \frac{1}{\left(1 - A\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)\right)} \frac{1}{\left(1 - \left(\frac{1}{2} - A\right)\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)\right)}$$

Among the particular cases of this family can be mentioned

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- When A = 0 or $A = \frac{1}{2}$ we obtain the Halley-type method to multiple roots (3.16).
- When $A = \frac{1 \pm \sqrt{2m+3}}{4}$ the following method is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{1 - \frac{1}{2}(1 - m + mL_f(x_n)) - \frac{m+1}{8}(1 - m + mL_f(x_n))^2} \right)$$
(3.26)

with $K(\alpha) = -\frac{A_2}{m}$

3.8 A new family mean of two Newton-Halley type methods to multiple roots

Let the weight function H given by

$$H(t) = \frac{1}{2 - At} + \frac{1}{2 - (2 - A)t}; \Rightarrow \ddot{H}(0) = \frac{1}{2}A^2 - A + 1$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{2 - A(1 - m + mL_f(x_n))} + \frac{1}{2 - (2 - A)(1 - m + mL_f(x_n))} \right)$$

with $K(\alpha) = \frac{1}{2m^2} \left[\left(m - 1 - 2A^2 + 4A \right) A_1^2 - 2mA_2 \right]$, which can be constructed from curve

$$x = x_n - m\frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \left(\frac{1}{2 - A\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)} + \frac{1}{2 - (2 - A)\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)}\right)$$

Among the particular cases of this family can be mentioned

- When A = 1 we obtain the Halley-type method to multiple roots (3.16).
- When A = 0 or A = 2 we obtain the Super-Halley-type method to multiple roots (3.17).
- When $A = 1 \pm \frac{\sqrt{2m+2}}{2}$ the following method is obtained

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{8 - 4(1 - m + mL_f(x_n))}{8 - 8(1 - m + mL_f(x_n)) + (1 - m)(1 - m + mL_f(x_n))^2} \right)$$
(3.27)

with $K(\alpha) = -\frac{A_2}{m}$.

3.9 An *A* family to multiple roots

Let the weight function H given by

$$H(t) = \frac{1}{1 - \frac{1}{2}t + At^2} \Rightarrow \ddot{H}(0) = \frac{(1 - 4A)}{2}$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \frac{1}{1 - \frac{1}{2}(1 - m + mL_f(x_n)) + A(1 - m + mL_f(x_n))^2}$$

with $K(\alpha) = \frac{1}{2m^2} \left[(m+1+8A)A_1^2 - 2mA_2 \right]$. This iteration equation can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \frac{1}{1 - \frac{1}{2} \left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) + A \left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)^2}$$

Among the particular cases of this family can be mentioned

- When A = 0 we obtain the Halley-type method to multiple roots (3.16).
- When $A = -\frac{m+1}{8}$ the method given by (3.26) is obtained.

3.10 A - C family to multiple roots

Let the weight function H be given by

$$H(t) = \frac{1}{1 - At} + \left(\frac{1}{2} - A\right)t + Ct^2 \Rightarrow \ddot{H}(0) = 2(A^2 + C)$$

So

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{1}{1 - A(1 - m + mL_f(x_n))} + \left(\frac{1}{2} - A\right) (1 - m + mL_f(x_n)) + C(1 - m + mL_f(x_n))^2 \right)$$

with $K(\alpha) = \frac{1}{2m^2} \left[\left(m + 3 - 8(A^2 + C) \right) A_1^2 - 2mA_2 \right]$. This iteration equation can be constructed from curve

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}} \right) \left(\frac{1}{1 - A\left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)} + \left(\frac{1}{2} - A\right) \left(1 - m + mL_f(x_n)\right) \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \right) + C \left(1 - m + mL_f(x_n)\right)^2 \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)^2 \right)$$

If C = 0 the family of the section 3.6 is obtained and if A = 0 we obtain the C-type family to multiple roots given by the equation (3.19).

3.11 A convex combination of two members of the family Newton-Halley to multiple roots ($A \neq B$)

Let the weight function H given by

$$H(t) = \frac{1}{2(A-B)} \left(\frac{1-2B}{(1-At)} + \frac{2A-1}{(1-Bt)} \right) \Rightarrow \ddot{H}(0) = A + B - 2AB$$

So

$$x_{n+1} = x_n - \frac{m}{2(A-B)} \frac{f(x_n)}{f'(x_n)} \left(\frac{1-2B}{\left(1 - A(1-m+mL_f(x_n))\right)} + \frac{2A-1}{\left(1 - B(1-m+mL_f(x_n))\right)} \right)$$

with $K(\alpha) = \frac{1}{2m^2} \left[(m+3-4(A+B-2AB))A_1^2 - 2mA_2 \right]$, which can be constructed from curve

$$x = x_n - \frac{m}{2(A-B)} \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \left(\frac{1 - 2B}{1 - A\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)} + \frac{2A - 1}{1 - B\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)}\right)$$

Among the particular cases of this family can be mentioned

- When $A = \frac{1}{2}$ or $B = \frac{1}{2}$ we obtain the Halley-type method to multiple roots (3.16).
- When A = 0 or B = 0 the family given in section 3.5 is obtained
- When $B = \frac{4A-m-3}{4(2A-1)}$ the following family is obtained

$$x_{n+1} = x_n - \frac{2m(2A-1)}{8A(A-1) + m + 3} \frac{f(x_n)}{f'(x_n)} \left(\frac{m+1}{2(2A-1)\left(1 - A(1-m+mL_f(x_n))\right)} + \frac{4(2A-1)^2}{(8A-4 - (m+3-4A)(1-m+mL_f(x_n)))} \right)$$

with $K(\alpha) = -\frac{A_2}{m}$

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4 Numerical examples

In order to evaluate the precision of the numerical schemes to solve the nonlinear scalar equations, we have applied several methods to four examples. To make such a comparison we select some classical methods for multiple roots and only a few methods presented in this work. The methods to be compared are the following: MNM: modified Newton's method (1.1), ChTM: Chebyshev-Type method (3.15), HTM: Halley-Type method (3.16), SHTM: Super-Halley Type method (3.17), OTM: Ostrowski-Type method (3.11), ETM: Euler-Type method (3.12) and ten methods presented in this paper: CM1: equation (3.13), CM2: equation (3.18), CM3: equation (3.20), CM4: equation (3.21), CM5: equation (3.22), CM6: equation (3.23), CM7: equation (3.24), CM8: equation (3.25), CM9: equation (3.26), CM10: equation (3.27).

Now the test functions and the approximations (x^*) to the multiple root α to be calculated are presented

$$f_{1}(x) = \left(e^{x^{2}+6x-16}-1\right)^{2} \left((x-1)^{3}-1\right)^{2} \qquad m=4 \qquad x^{*}=2,$$

$$f_{2}(x) = \left(xe^{x^{2}}-\sin^{2}x+3\cos x+5\right)^{4} \qquad m=4 \qquad x^{*}=-1.20764782713\cdots,$$

$$f_{3}(x) = \left(e^{x^{2}+4x+5}-1\right)^{3} \left(\sinh(2+i+ix)\right)^{2} \qquad m=5 \qquad x^{*}=-2+i$$

$$f_{4}(x) = \left(x^{2}-1\right)^{2} \qquad m=2 \qquad x^{*}=1$$

The first three functions are selected from [13].

In order to evaluate the accuracy of numerical schemes the computational order of convergence (COC) is used, which is given by [17]

$$COC = \frac{\ln |(x_{n+1} - x^*)/(x_n - x^*)|}{\ln |(x_n - x^*)/(x_{n-1} - x^*)|}$$

A comparison of several iterative methods under the same total number of function evaluations (TNFE = 12) is presented in Tables 1 and 2. Thus, $|x_n - x^*|$ is an approximation to the absolute error, where n = 4 in all methods of third order and n = 6 for Modified Newton's method; $|f_i(x_n)|, i = 1, 2, 3, 4$ is the absolute value of the test function f_i in x_n . The computational order of convergence (*COC*) is also presented. Representative values of a very good or very bad behavior are highlighted in bold.

In Table 1 for the case of the test function f_1 , $x_0 = 1.7$ is used; you can see that CM7 and CM10 methods diverge, CM2 converges very slowly and CM4 is the one with the better behavior; all other methods have an acceptable behavior. In the case of the test function f_2 , $x_0 = -1.5$ is used; you can see that CM2 converges initially very slowly, CM10 diverges and CM8 is the method with the better behavior. All other methods have a good behavior.

Table 1: Comparison of various iterative methods under the same total number of function evaluations (TNFE=12). $x_0 = 1.7$ for $f_1(x)$ and $x_0 = -1.4$ for $f_2(x)$.

	$ x_n - x^* $	$ f_1(x_n) $	COC	$ x_n - x^* $	$ f_2(x_n) $	COC
MNM	1.0934069e-12	1.2863816e-45	1.9998944	4.3357831e-26	6.0101938e-97	2,0000000
ChTM	5.3691108e-14	7.4791456e-51	2.9941953	4.2922336e-31	5.7723367e-117	2.9999914
HTM	9.9433399e-27	8.7977507e-102	3.0001408	3.5352693e-57	2.6564906e-221	3.0000001
SHTM	1.2061119e-05	1.9048327e-17	3.0663800	2.9486859e-25	1.2856784e-93	3.0001866
OTM	3.2451414e-28	9.9810677e-108	3.0000248	3.3715917e-41	2.1976482e-157	3.0000021
ETM	1.9432007e-08	1.2832554e-28	3.0975287	3.5465646e-18	2.6906039e-65	3.0975287
CM1	4.0952912e-05	2.5314844e-15	3.5488957	5.9258889e-15	2.0971634e-52	3.0148851
CM2	0.24352993e00	105.27655	0.6144833	0.36719692e00	48242.606	0.2983138
CM3	1.5365968e-27	5.0174403e-105	3.0002087	5.0980796e-33	1.1488008e-124	2.9999740
CM4	5.1569205e-34	6.3650859e-131	3.0000359	7.2167449e-44	4.6130025e-168	2.9999991
CM5	4.9010838e-20	5.1929129e-75	2.9989211	1.6820526e-22	1.3613714e-82	2.9997901
CM6	1.5070996e-07	4.6431325e-25	2.8641060	2.2986402e-18	4.7479075e-66	2.9973513
CM7	48.259876e00	1.9594700e2452	-3.8839742	1.4081354e-12	6.6864549e-43	3.0025125
CM8	7.0197164e-21	2.1853488e-78	3.0028502	8.9545700e-62	1.0934457e-239	3.0000000
CM9	1.1106763e00	3.6945899e12	0.8074724	1.7997228e-08	1.7841886e-26	3.0093347
CM10	1.3270085e00	1.5302388e15	0.8242423	2.4085957e00	8.9224116e24	0.3610180

When the test function f_3 is used, $x_0 = -2.2 + 1.2I$ in Table 2. Here, CM6 has the greatest absolute error, CM1, ETM, CM2 and SHTM have excellent behavior and the others methods have a very good behavior. For the case of the test function f_4 , $x_0 = 1.5$ is used. It is highlighted that SHTM and CM4 have fourth order of convergence for this example, CM7 and CM9 have very good behavior. All other methods have a good behavior. It is noteworthy that in all tests Modified Newton's method has a good behavior.

Table 2: Comparison of various iterative methods under the same total number of function evaluations (TNFE=12). $x_0 = -2.2 + 1.2I$ for $f_3(x)$ and $x_0 = 1.5$ for $f_4(x)$.

	$ x_n - x^* $	$ f_3(x_n) $	COC	$ x_n - x^* $	$ f_4(x_n) $	COC
MNM	3.2015327e-64	2.6907894e-317	2.0000000	3.6893488e-45	5.4445179e-89	2.0000000
ChTM	6.8275183e-61	1.1868733e-300	2.9594112	6.686974e-48	1.7886249e-94	2.9999993
HTM	2.5332660e-70	8.3462997e-348	3.0000000	4.8357033e-57	9.3536105e-113	2.9999999
SHTM	4.4106969e-80	1.3354451e-396	3.0000000	2.9999998e-179	2.1452497e-357	4.0000000
OTM	4.9390126e-76	2.3512058e-376	3.0000000	7.5233794e-67	2.2640495e-132	3.0000000
ETM	1.3504131e-81	3.5927186e-404	3.0000000	0.	0.	
CM1	6.8540098e-86	1.2100787e-425	3.0000000	6.5684306e-59	1.7257712e-116	3.0000000
CM2	2.1184516e-80	3.4133648e-398	3.0000000	8.0237702e-74	2.5752356e-146	2.7393875
CM3	3.1491970e-70	2.4779307e-347	2.9999998	6.1322400e-74	1.5041747e-146	3.0000000
CM4	3.8320518e-76	6.6106911e-377	3.0000000	1.4926985e-148	8.9125948e-296	4.0000000
CM5	1.4369018e-72	4.9003317e-359	2.9999998	1.1637299e-84	5.4170693e-168	3.0000000
CM6	4.1735425e-44	1.0130111e-216	2.9999974	3.0202954e-38	3.6488738e-75	2.9999878
CM7	4.5472528e-75	1.5553763e-371	2.9999999	3.6151703e-108	5.2277824e-215	3.0000000
CM8	2.7680440e-66	1.3000348e-327	3.0000004	3.637195e-68	5.2916763e-135	3.0000000
CM9	1.4001416e-74	4.3047686e-369	2.9999999	1.0293706e-97	4.2384155e-194	3.0000000
CM10	2.0893021e-76	3.1849028e-378	3.0000000	1.1354528e-78	5.1570123e-156	3.0000000

In the next two subsections, we do a comment on the application of the Euler-type method for multiple roots when it is applied to the function f_4 since in this case in an iteration the exact solution is obtained.

4.1 A Geometric constrution for ETM with f_4

Using a = 1 in the equation of the given curve in (3.10) we obtain the expression that allows constructing the Euler-type method for multiple roots

$$x = x_n - m \frac{f(x_n)}{f'(x_n)} \left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right) \frac{2}{1 + \sqrt{1 - 2\left(1 - m + mL_f(x_n)\right)\left(1 - \sqrt[m]{\frac{y}{f(x_n)}}\right)}}$$
(4.28)

For test function f_4 we have to

$$f(x) = (x^{2} - 1)^{2} \qquad \Rightarrow f'(x) = 4x(x^{2} - 1)$$

$$\Rightarrow f''(x) = 4(3x^{2} - 1) \Rightarrow L_{f}(x) = \frac{(3x^{2} - 1)}{4x^{2}}$$
(4.29)

As m = 2, then

$$m\frac{f(x)}{f'(x)} = \frac{x^2 - 1}{2x}$$
 and $1 - m + mL_f(x) = \frac{x^2 - 1}{2x^2}$. (4.30)

Substituting (4.29) and (4.30) into (4.28), the following curve is obtained

$$x = x_n - \frac{|x_n|(x_n^2 - 1)}{x_n |x_n^2 - 1|} \frac{|x_n^2 - 1| - \sqrt{y}}{|x_n| + \sqrt{1 + \frac{x_n^2 - 1}{|x_n^2 - 1|}\sqrt{y}}}$$
(4.31)

The following cases to analyze the curve given by (4.31) are presented.

• If in (4.31) we consider $x_n < -1$, then

$$x = \frac{1 - x_n \sqrt{1 + \sqrt{y}} + \sqrt{y}}{x_n - \sqrt{1 + \sqrt{y}}} \Rightarrow y = \left(\frac{(x_n - x + |x + x_n|)^2}{4} - 1\right)^2,$$

now, if $x + x_n < 0$ then $y = (x^2 - 1)^2 = f(x)$.

• If in (4.31) we consider $-1 < x_n < 0$, then

$$x = \frac{1 - x_n \sqrt{1 + \sqrt{y}} - \sqrt{y}}{x_n - \sqrt{1 + \sqrt{y}}} \Rightarrow y = \left(\frac{(x_n - x + |x + x_n|)^2}{4} - 1\right)^2$$

again, if $x + x_n < 0$ then $y = (x^2 - 1)^2 = f(x)$.

• If in (4.31) we consider $0 < x_n < 1$, then

$$x = \frac{1 + x_n \sqrt{1 - \sqrt{y}} - \sqrt{y}}{x_n + \sqrt{1 - \sqrt{y}}} \Rightarrow y = \left(\frac{(x - x_n + |x + x_n|)^2}{4} - 1\right)^2$$

if $x + x_n > 0$ then $y = (x^2 - 1)^2 = f(x)$.

• If in (4.31) we consider $x_n > 1$, then

$$x = \frac{1 + x_n \sqrt{1 + \sqrt{y}} + \sqrt{y}}{x_n + \sqrt{1 + \sqrt{y}}} \Rightarrow y = \left(\frac{(x - x_n + |x + x_n|)^2}{4} - 1\right)^2$$

In the same way as in the previous item, if $x + x_n > 0$ then $y = (x^2 - 1)^2 = f(x)$.

• Finally, if $x_n = 0$, since f'(0) = 0 a curve that meets the conditions of tangency can not be constructed. If $x_n = \pm 1$ it is not necessary to construct a curve, these values are roots of *f*.

In all the cases the same function to which it was desired to calculate the roots was obtained.

4.2 Iteration equation for ETM with f_4

Applying the Euler-type method for multiple roots (3.12) to f_4 gives the following iteration

$$x_{n+1} = \frac{x_n^2 + |x_n|}{x_n(|x_n| + 1)}$$

If $x_n > 0$ then $x_{n+1} = 1$. So, using $x_0 > 0$ then $x_1 = 1$. If $x_n < 0$ then $x_{n+1} = -1$. So, using $x_0 < 0$ then $x_1 = -1$.

In this way, it is concluded that when applying the Euler-type method for multiple roots (3.12) to $f_4(x) = (x^2 - 1)^2$, the solution of the nonlinear equation $f_4(x) = 0$ is obtained in a single iteration.

5 Final Remarks

We have developed a new form to obtain a geometric construction of methods of third order to solve a nonlinear equation with multiple roots when the multiplicity is known. This construction can be applied to any method obtained from the class given in [13]. The curve obtained satisfies the conditions of tangency $y(x_n) = f(x_n)$, $y'(x_n) = f'(x_n)$ and $y''(x_n) = f''(x_n)$. Also, several new families and/or third order methods to solve nonlinear equations and their geometric construction are presented. Then, the precision of some numerical schemes obtained to solve the nonlinear scalar equations has been analyzed from four examples. Of course, it is necessary to do more analysis on these families, especially from a dynamic point of view.

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References

- [1] J. F. Traub, Iterative methods for resolution of equations, Prentice Hall, NJ, (1964).
- [2] A. M. Ostrowski, Solution of equations and systems of equations, Prentice-Hall, Engle-wood Cliffs, NJ, USA, (1964).
- [3] A. M. Ostrowski, Solution of Equations in Euclidean and Banach Space, Academic Press, New York, (1973).
- [4] W. Gander, On Halley's iteration method, Am Math Mon, 92 (1985) 131-134. https://doi.org/10.2307/2322644
- [5] T. R. Scavo, J. B. Thoo, On the geometry of Halley's method, Am Math Mon, 102 (1995) 417-426. https://doi.org/10.2307/2975033
- [6] A. Melman, Geometry and convergence of Euler's and Halley's methods, SIAM Rev.39, 4 (1997) 728-735. https://doi.org/10.1137/S0036144595301140
- [7] S. Amat, S. Busquier, J. M. Gutiérrez, Geometric constructions of iterative functions to solve nonlinear equations, J Comput Appl Math, 157 (2003) 197205. https://doi.org/10.1016/S0377-0427(03)00420-5
- [8] V. Kanwar, Sukhjit Singh, S. Bakshi, Simple geometric constructions of quadratically and cubically convergent iterative functions to solve nonlinear equations, Numer Algor, 47 (2008) 95-107. https://doi.org/10.1007/s11075-007-9149-4
- [9] R. Behl, S. S. Motsa, Geometric Construction of Eighth-Order Optimal Families of Ostrowski's Method, The Scientific World Journal, (2015). https://doi.org/10.1155/2015/614612



- [10] C. E. Cadenas, On several Gander's theorem based third order iterative methods for solving nonlinear equations and their geometric constructions, J. Numer. Math. Stoch, 9 (1) (2017) 1-18.
- [11] E. Schröder, Über unendlichviele Algorithm zur Auffosung der Gleichungen, Math. Ann, 2 (1870) 317-365. https://doi.org/10.1007/BF01444024
- [12] C. E. Cadenas. Some geometric constructions of two variants of Newton's method to solving nonlinear equations with multiple roots. Punjab University Journal of Mathematics. 50, 1 (2018) 15-21.
- [13] I. Petković, D. Herceg, Symbolic computation and computer graphics as tools for developing and studying new root-finding methods, Appl. Math. Comput, 295 (2017) 95-113. https://doi.org/10.1016/j.amc.2016.09.025
- [14] E. Hansen, M. Patrick, A family of root finding method, Numer. Math, 27 (1977) 257-269. https://doi.org/10.1007/BF01396176
- [15] J. M. Gutiérrez, M. A. Hernández, A family of Chebyshev-Halley type methods in Banach spaces, Bull. Austral. Math. Soc, 55 (1997) 113-130. https://doi.org/10.1017/S0004972700030586
- S. Amat, S. Busquier, S. Plaza, On the dynamics of a family of third-order iterative functions, ANZIAM J, 48 (2007) 343-359. https://doi.org/10.1017/S1446181100003539
- [17] S. Weerakoon, T. G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett, 13 (8) (2000) 87-93. https://doi.org/10.1016/S0893-9659(00)00100-2